found for values of $\varphi_L \leq 0.19$ rad. Up to the present, it has not been possible to find such solutions.

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Prestressing in Structural Synthesis

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Introduction

PRESTRESSING is a recognized device for improving structural efficiency. The most widespread applications are found in concrete construction, where precompression is introduced into structures in order to limit or eliminate tenile stresses and consequent brittle failure. The immediate effect of such prestressing is usually a reduction in weight and/or deflection compared to that of nonprestressed structures.

Application of prestressing to structures composed of other than brittle materials occurs relatively infrequently, even in aerospace applications where the potential enhancement of high-performance structures could prove significant. This situation is probably attributable both to the lack of a comprehensive approach to optimization of prestressed structures and to fabrication difficulties.

This Note attempts to clarify the first area by describing a general method for including prestressing in the minimum-weight design of statically indeterminate structures. The design problem is formulated as a constrained minimization problem, and the magnitude of the prestress, as well as the usual parameters, are considered as variables which must be selected optimally.

Synthesis

It has been shown[1] that the minimum-weight structural design problem can be cast in the form of a mathematical programing problem as follows:

$$\begin{aligned} &\text{Minimize } f(\mathbf{X}) \text{ over } \mathbf{X} \\ &\text{such that } g_k(\mathbf{X}) \geq 0 \text{ for all } k \end{aligned} \qquad (1)$$

where $\mathbf{X} = \{x_1\, x_2 \ldots x_n\}$ is a vector of design variables, $f(\mathbf{X})$ is the objective function, and the $g_k(\mathbf{X})$ are constraint functions. The design variables may be, for example, cross-sectional areas of members in truss design, and so on. The objective function is taken as the total weight of the structural system, and the constraints limit structural behavior. For example, the constraints may specify that member stresses do not exceed limiting values, or that deflections remain within specified bounds. Equations (1) can be solved numerically by searching systematically over the acceptable values of $\mathbf{X}$ [i.e., those $\mathbf{X}$ for which $g_k(\mathbf{X}) \geq 0$] for the minimum of the objective function. In general, $f(\mathbf{X})$ and the $g_k(\mathbf{X})$ are nonlinear functions of the x_i.

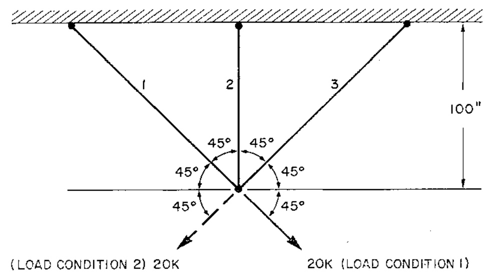

Fig. 1 Truss design example.

Consider now that prestressing can be introduced either by the use of high-strength tension cables as in concrete structures, or alternately could be induced by an initial lack of fit of the structural members if the system is indeterminate. In either case let the vector of original design variables $\mathbf{X}$ be augmented by $x_{n+1}, \ldots, \ldots, x_m$, where the additional variables represent unknown quantities associated with the prestressing. For example, with cable prestressing, the new variables will represent initial values of precompression for each component of the system. If the system is initially stressed by lack of fit, the additional design variables may be taken as the changes in length of each member necessary to achieve the prestress. The augmented vector $\{x_1, \ldots, x_{n+1}, \ldots, x_m\}$ will be denoted by $\hat{\mathbf{X}}$. It should be noted that the initially prestressed structure must satisfy all conditions of displacement compatibility.

If it is assumed that the prestressing mechanism does not contribute significantly to the total structural weight, then the objective function remains dependent only on $\mathbf{X}$ rather than $\hat{\mathbf{X}}$. The effect of the prestressing on the design problem then enters by way of the constraint equations. For example, if a structural analysis reveals that the stress in member i under load condition j is $\sigma_{ij}(\mathbf{X})$, then the stress constraints can take the normalized form

$$g_k(\mathbf{X}) = [\sigma_y - \sigma_{ij}(\mathbf{X})]/\sigma_y \geq 0 \qquad (2)$$

where σ_y may be a yield stress of the material or other allowable stress limit. For a linearily responding structural system, the prestressing modifies the stress distribution to $[\sigma_{ij}(\mathbf{X}) + \hat{\sigma}_i(\hat{\mathbf{X}})]$ where the $\hat{\sigma}_i(\hat{\mathbf{X}})$ is caused by the prestress. (Since this additional stress is a constant for all loading conditions, the second subscript has been dropped.) The constraints on stress behavior now become

$$\hat{g}_k(\hat{\mathbf{X}}) = [\sigma_y - \sigma_{ij}(\mathbf{X}) - \hat{\sigma}_i(\hat{\mathbf{X}})]/\sigma_y \geq 0 \qquad (3)$$

In addition to the modified stress constraints, Eq. (3), the structure must now withstand the initial stresses induced by prestress before any load is applied. Thus, supplementary stress constraints of the form

$$\hat{g}_p(\hat{\mathbf{X}}) = [\hat{\sigma}_y - \hat{\sigma}_i(\hat{\mathbf{X}})]/\hat{\sigma}_y \geq 0 \qquad (4)$$

are necessary. Here $\hat{\sigma}_y$ may or may not equal σ_y. Deflection or other behavioral constraints can be modified in a similar manner.

The new mathematical programing problem can now be stated in a general form as

$$\min_{\hat{\mathbf{X}}} f(\mathbf{X}) \text{ such that } \hat{g}_k(\hat{\mathbf{X}}) \geq 0 \text{ for all } k \qquad (5)$$

Examples

In order to illustrate the preceding method, the planar truss of Fig. 1 will be considered. This example was first

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Table 1 Truss design—nonprestressed case

Member	Area (in.²)	Stress, Load 1 (ksi)	Stress, Load 2 (ksi)
1	0.797	19.76*	−5.33
2	0.416	14.44	14.44
3	0.797	−5.33	19.76*

presented by Schmit,[1] and contains many of the characteristics of the general minimum-weight structural design problem.

The truss has the geometry specified, and is subjected to the two independent load conditions indicated in the figure. The behavior is assumed to be of the linearly elastic-small deflection type, and buckling of the individual elements is not considered. For the nonprestressed case, the design variables are the cross-sectional areas of the three components. The objective function is linear, and twelve nonlinear behavioral constraints can be generated by a requirement that -15 ksi $\leq \sigma_{ij} \leq 20$ ksi; i.e., $\sigma_y = 20$ ksi for tension and 15 ksi for compression. Three side constraints specifying that the design variables be non-negative are also required.

The results of the optimum design of this structure are shown in Table 1. Using a material density of 0.01 lb/in.³, this design corresponds to a weight of 2.670 lb. Note that only two constraints are active for this design: those on yielding in tension of members 1 and 3 in loading condition 1 and 2, respectively. These critical stresses are noted in Table 1 with asterisks.

If the components of the truss of Fig. 1 are now assumed to be each precompressed by tension cables, three more design variables must be considered, namely the prestress in members 1, 2, and 3. Thus, the constraints of the preceding example must be modified as in Eq. (3), and six additional constraints of the form of Eq. (4) are required. These latter constraints state that the prestresses must be bounded by -15 ksi $\leq \hat{\sigma}_i \leq 0$ since positive prestressing is not possible.

The results of this design are shown in Table 2. Weight is equal to 1.796 lb which constitutes a 33% reduction from the previous case. In this design, four constraints are binding at the optimum.

For the final example, consider the truss of Fig. 1 prestressed by an initial lack of fit of the three components. The additional design variables are now the changes in length (from nominal) of each member necessary to prestress the structure. For this case $\hat{\sigma}_y$ is taken equal to σ_y in both compression and tension.

Table 3 shows the final design, which has a weight of 2.226 lb. This design is fully stressed in both loading conditions and is 9% lighter than the optimal nonprestressed truss of Table 1. The existence of such a fully stressed design, made feasible by prestressing, was indicated in Ref. 2.

All numerical results presented above were obtained using an algorithm suggested by Fiacco and McCormick,[3] which transformed the problem of Eq. (5) into an unconstrained minimization problem. The penalty function $P(\mathbf{X},r_p)$ was minimized for each unconstrained cycle by the Variable Metric Method.[4,5] The initial value r_1 was computed using criterion 1 of Ref. 3 and subsequent values of r_p were calculated from $r_{p+1} = r_p/2$. For the example of Table 1, thirteen unconstrained cycles were executed; for design 2, ten cycles; and for design 3, twenty-one cycles. Design 2 was stopped somewhat short of full convergence. All examples used as the initial design $A_1 = A_3 = 2.0, A_2 = 1.0$ and zero prestress.

Table 2 Truss design—prestressed with cables

Member	Area, in.²	Prestress, ksi	Stress, load 1, ksi	Stress, load 2, ksi
1	0.609	−11.55	18.88*	−14.00*
2	0.075	−11.01	16.98	16.98
3	0.609	−11.55	−14.00*	18.88*

Table 3 Truss design—prestressed by lack of fit

Member	Area, in.²	Change in length, in.²	Prestress, ksi	Stress, load 1, ksi	Stress, load 2, ksi
1	0.573	0.0596	−4.42	19.98*	−14.96*
2	0.606	−0.0651	5.98	19.96*	19.96*
3	0.573	0.0596	−4.42	−14.96*	19.98*

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Cross-Hatching: A Material Response Phenomena

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Model

One of the most intriguing and least understood fluid mechanical problems to have arisen over the past several years is that of "cross-hatching."[1–4] The term crosshatching refers to the spatially fixed diamond shaped patterns which have been observed to form on the surfaces of ablating bodies exposed to a supersonic turbulent or transitional boundary-layer flow. That the crosshatching appears on ablating surfaces has led to the idea the patterns might result from an interaction between the ablated material and the external supersonic flow (see, e.g., Ref. 4). Since some ablating materials melt, Nachtsheim[5] has also suggested that surface tension may be responsible for the formation of the patterns. We believe that ablation per se is not responsible for the origin of the cross-hatching but rather that the observed characteristic patterns result from the fact that materials which ablate become inelastic deformable relaxing and creeping bodies prior to ablating.

The model considered here is the response of an anelastic surface to the pressure fluctuations induced by a supersonic turbulent boundary layer leading to the cross-hatching as a consequence of a differential deformation due to a relaxation within the material. The patterns are not initiated as a result of differential ablation and can form in the absence of ablation. The pattern wavelength imposed on the material by the gaseous boundary layer λ must be of the order

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